On Total and Edge-colouring of Proper Circular-arc Graphs

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Abstract. Deciding if a graph is $\Delta$-edge-colourable (resp. $(\Delta+1)$-total colourable), although it is an NP-complete problem for graphs in general, is polynomially solvable for interval graphs of odd (resp. even) maximum degree $\Delta$. An interesting superclass of the proper interval graphs are the proper circular-arc graphs, for which we suspect that $\Delta$-edge-colourability is linear-time decidable. This work presents sufficient conditions for $\Delta$-edge-colourability, $(\Delta+1)$-total colourability, and $(\Delta+2)$-total colourability of proper circular-arc graphs. Our proofs are constructive and yield polynomial-time algorithms.

1. Introduction

The chromatic index and the total chromatic number of a graph $G$ with maximum degree $\Delta$ clearly satisfy $\chi'(G) \geq \Delta$ and $\chi''(G) \geq \Delta + 1$ (see definitions in the sequel). Also, $\chi'(G) \leq \Delta + 1$ [Vizing 1964], and the Total Colouring Conjecture states that $\chi''(G) \leq \Delta + 2$ [Behzad 1965, Vizing 1968]. A graph $G$ is Class 1 if $\chi'(G) = \Delta$, or Class 2 otherwise. Since no graph with $\chi''(G) \geq \Delta+3$ is known, graphs with $\chi''(G) = \Delta+1$ have been called Type 1, and those with $\chi''(G) = \Delta + 2$ Type 2. Deciding if $G$ is Class 1 and deciding if $G$ is Type 1 are NP-complete problems [Holyer 1981, Sánchez-Arroyo 1989].

The classes of the unit and the proper interval graphs are the same [Roberts 1969], but the classes of the unit and the proper circular-arc graphs are not (see Figure 1). The

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A proper non-unit circular-arc graph with a corresponding arc model}
\end{figure}

Total Colouring Conjecture holds for proper interval graphs, often referred to as indifference graphs, which are Class 1 when they have odd $\Delta$, and Type 1 when $\Delta$ is even [Figueiredo et al. 1997]. For edge-colouring of indifference graphs with even $\Delta$ or total colouring of these graphs with odd $\Delta$, partial results are known [Figueiredo et al. 2003, Partly supported by CNPq, Proc. 428941/2016-8. Partly supported by UFFS, Proc. 23205.001243/2016-30.}
Let $G$ be an $n$-vertex proper circular-arc graph. We show that if $n \equiv 0 \pmod{\Delta+1}$, or if $G$ has a maximal clique of size 2 and $n \not\equiv k \pmod{\Delta+1}$ for all $k \in \{1, \Delta\}$, then: $\chi(G) = \Delta$ and $\chi''(G) \leq \Delta + 2$ if $\Delta$ is odd; $\chi''(G) = \Delta + 1$ if $\Delta$ is even. This implies that the Total Colouring Conjecture holds for the class of all such graphs.

This paper is organised as follows: the remaining of this section provides further definitions and discusses other related results in the literature; Section 2 presents our results; at last, Section 3 makes remarks on edge-colouring proper circular-arc graphs.

Preliminary definitions and other related results

This work deals only with simple graphs, referred to simply as graphs. Usual terms concerning graph-theoretical concepts follow their definitions and notation in the literature. In particular, the degree of a vertex $u$ in a graph $G$, the set of neighbours of $u$ in $G$, and the set of the edges incident to $u$ in $G$ are denoted by $d_G(u)$, $N_G(u)$, and $\partial_G(u)$, respectively.

Let $G = (V,E)$ be a graph and $\mathcal{C}$ a set of $t$ colours. A function with $\mathcal{C}$ as its codomain is: a $t$-edge-colouring if its domain is $E$ and it is injective in $\partial_E(v)$ for all $v \in V$; a $t$-total colouring if its domain is $V \cup E$ and it is injective in $\partial_G(u) \cup \{u\}$ for all $u \in V$ and injective in $\{u,v\}$ for all $uv \in E$. In a total or edge-colouring, we say that a colour is missing at a vertex $u$ if it is not assigned to $u$ or to any edge incident to $u$. The chromatic index (denoted by $\chi'(G)$) and the total chromatic number (denoted by $\chi''(G)$) of $G$ are the least $t$ for which $G$ is $t$-edge-colourable and $t$-total colourable, respectively.

An $n$-vertex graph with more than $\Delta \lfloor n/2 \rfloor$ edges (thus Class 2, since at most $\lfloor n/2 \rfloor$ edges can be coloured the same) is said to be overfull. It is conjectured that every graph $G$ with $\Delta > n/3$ is Class 2 if and only if it is subgraph-overfull (shortly, SO), i.e. if $G$ has an overfull subgraph with the same maximum degree [Hilton and Johnson 1987].

The complete graph $K_n$ is: Class 1 and Type 2 if $n$ is even; Class 2 and Type 1 if $n$ is odd [Behzad et al. 1967]. Let $V(K_n) := \{0, \ldots, n-1\}$ and let even($n$) be 1 if $n$ is even or 0 otherwise. We call the canonical total and edge-colourings of the $K_n$ the functions $\varphi_{\text{even}}$, $\varphi_{\text{odd}}$, and $\varphi_{\text{total}}$ given by:

<table>
<thead>
<tr>
<th>$uv$</th>
<th>$\varphi_{\text{even}}(uv)$</th>
<th>$\varphi_{\text{odd}}(uv)$</th>
<th>$\varphi_{\text{total}}(uv)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>$(u+v) \pmod{(n-1)}$</td>
<td>$(u+v) \pmod{(n+\text{even}(n))}$</td>
<td>$(u+v) \pmod{(n+\text{even}(n))}$</td>
</tr>
<tr>
<td>odd</td>
<td>$(2u) \pmod{(n-1)}$</td>
<td>$(2u) \pmod{(n+\text{even}(n))}$</td>
<td>$(2u) \pmod{(n+\text{even}(n))}$</td>
</tr>
</tbody>
</table>

A circular-arc graph $G$ is the intersection graph of a finite set $S$ of arcs of a circle, in which case $S$ is an arc model corresponding to $G$. Furthermore, $G$ is said to be: proper, if there is a corresponding arc model wherein no arc properly contains another; a unit circular-arc graph, if there is a model wherein all the arcs have equal length. The vertices of a proper circular-arc graph admit a proper circular-arc order, i.e. a circular order in which vertices belonging to the same maximal clique appear consecutively. Homonymous terms are defined for interval graphs analogously, but being $S$ a set of intervals on the real line and the proper interval (or indifference) order a linear order. Interval and circular-arc graphs can be recognised in linear time [Booth and Lueker 1976, McConnell 2003].

A pullback from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ is a homeomorphism $\pi: V_1 \to \pi(V_1)$.
V_2 \text{ (i.e. } \pi(u) \pi(v) \in E_2 \text{ for all } uv \in E_1) \text{ injective in } N_{G_1}(u) \cup \{u\} \text{ for all } u \in V_1. \text{ If such a pullback exists and } G_2 \text{ has: } t\text{-edge-colouring } \varphi, \text{ then a } t\text{-edge-colouring for } G_1 \text{ can be given by } \psi(uv) := \varphi(\pi(u) \pi(v)); \text{ } t\text{-total colouring } \varphi, \text{ then a } t\text{-total colouring for } G_1 \text{ can be given by } \psi(uv) := \varphi(\pi(u) \pi(v))\text{ and } \psi(u) := \varphi(\pi(u)) \text{ [Figueiredo et al. 1997].}

2. Results

Throughout this section, let G be an n-vertex proper circular-arc graph. Remark that when we say that G is \(\Delta + 2\)-total colourable, it does not mean that G is Type 2.

**Theorem 1.** If \(n \equiv 0 \pmod{\Delta + 1}\), then G is: Class 1 and \((\Delta + 2)\)-total colourable if \(\Delta\) is odd; Type 1 if \(\Delta\) is even.

**Proof.** It suffices to show that if \(n \equiv 0 \pmod{\Delta + 1}\), then there is a pullback from G to the \(K_{\Delta+1}\). Let \(\sigma := u_0, \ldots, u_{n-1}\) be a proper circular-arc order of G and \(0, \ldots, \Delta\) be the vertices of the \(K_{\Delta+1}\). Assume, by the sake of contradiction, that the function \(\pi: V(G) \to V(K_{\Delta+1})\) defined by \(\pi(u_i) := i \mod{\Delta + 1}\) is not a pullback from G to the \(K_{\Delta+1}\). As \(\pi\) is clearly a homeomorphism, there must be two distinct vertices \(v_1\) and \(v_2\) in \(V(G)\) which have a neighbour \(w\) in common and satisfy \(\pi(v_1) = \pi(v_2)\). However, since \(\sigma\) is a proper circular-arc order of G, all vertices between \(v_1\) and \(v_2\) in \(\sigma\) are thus neighbours of \(w\), which straightforwardly implies \(d_G(w) > \Delta\).

**Theorem 2.** If \(n \not\equiv k \pmod{\Delta + 1}\), for all \(k \in \{1, \Delta\}\), and G has a maximal clique of size 2, then G is: Class 1 and \((\Delta + 2)\)-total colourable if \(\Delta\) is odd; Type 1 if \(\Delta\) is even.

**Proof.** If \(r := n \mod{\Delta + 1} = 0\), we are done by Theorem 1. If \(\Delta \leq 2\), then G is a cycle or a disjoint union of paths and the theorem clearly holds. Hence, we assume that \(\Delta \geq 3\) and \(r \not= 0\). Let \(\sigma := u_0, \ldots, u_{n-1}\) be a proper circular-arc order of G, being \(\{u_0, u_{n-1}\}\) a maximal clique. Because \(\sigma\) is a proper circular-arc order, we have \(u_\Delta \not\in N_G(u_0)\) and \(u_{n-1-\Delta} \not\in N_G(u_{n-1})\), otherwise \(d_G(u_0) > \Delta\) or \(d_G(u_{n-1}) > \Delta\).

Let \(V(K_{\Delta+1}) := \{0, \ldots, \Delta\}\) and let \(\varphi \in \{\varphi_{\text{even}}; \varphi_{\text{total}}\}\) be the canonical total or edge-colouring of the \(K_{\Delta+1}\). The function \(\pi: V(G') \to V(K_{\Delta+1})\) defined by \(\pi(u_i) := i \mod{\Delta + 1}\) is clearly a homeomorphism. The function \(\psi: V(G') \to V(G)\) defined by \(\psi(v_i) := \varphi(\pi(u_i))\) is a pullback from \(G' := G - u_{n-1}u_0\) to the \(K_{\Delta+1}\) and brings a total or an edge-colouring \(\psi\) of \(G'\) using the same set of colours as \(\varphi\). Ergo, we have only to colour \(u_{n-1}u_0\) in order to complete the proof.

Observe that \(\pi(u_{n-1}) = r-1, \pi(u_{n-1-\Delta}) = r, \text{ and, since neither } r \text{ nor } r-1 \text{ is } \Delta, \varphi(r, r-1) = (2r-1) \pmod{\Delta} = q, \text{ with } d := \Delta \text{ if } \varphi = \varphi_{\text{edge}}, \text{ or } d := \Delta + 1 + \text{even}(\Delta + 1) \text{ if } \varphi = \varphi_{\text{total}}. \text{ Therefore, as } \pi(v) \not= \Delta \text{ and } \pi(w) \not= r \text{ for all } v \in N_G(u_0) \text{ and all } w \in N_G(u_{n-1}), \text{ the colour } \varphi(0, \Delta) \text{ is missing at } u_0 \text{ and the colour } q \text{ at } u_{n-1}. \text{ If } q = \varphi(0, \Delta), \text{ then we assign the colour } q \text{ to } u_{n-1}u_0 \text{ and we are done. Otherwise, since } q \in \{0, \ldots, \Delta\}, \text{ we exchange } \Delta \text{ and } q \text{ in the codomain of } \pi, \text{ that is, we redefine } \pi \text{ so that every vertex which has been mapped by } \pi \text{ to } \Delta \text{ is now mapped to } q \text{ and vice versa. Notice that the images of } u_0, u_{n-1-\Delta}, \text{ and } u_{n-1} \text{ by } \pi \text{ remain the same, but } \pi(u_\Delta) \text{ becomes } q, \text{ which now is also a colour missing at } u_0. \text{ Then, we colour } u_{n-1}u_0 \text{ with } q. \Box

3. Final remarks

Let \(\mathcal{A}\) be the class of the proper circular-arc graphs with odd \(\Delta\) and a maximal clique of size 2. Overfull graphs in \(\mathcal{A}\) can be constructed for \(n \equiv 1\) and for \(n \equiv \Delta \pmod{\Delta + 1}\) (see Figures 2(a) and 2(b), respectively). Since Theorem 2 can be interestingly used to
show a graph in $A$ is SO if and only if it is overfull, we conclude proposing the following:

**Conjecture.** A graph in $A$ is Class 2 if and only if it is overfull.

**References**


